Note

Fast Direct Poisson Solvers for High-Order Finite Element Discretizations in Rectangularly Decomposable Domains

INTRODUCTION

It has been recognized for some time that substructure (or subdomain) methods can significantly increase the efficiency of finite element solution of elliptic partial differential equations [1]. In substructuring techniques, the computational domain, D, is broken up into N_S subdomains, E_k, $D = \bigcup_k E_k$. Block elimination is then used to reduce the finite element equations to N_S sets of calculations for the internal degrees-of-freedom in the individual subdomains, coupled by a statically-condensed system matrix involving only the nodes on substructure boundaries. This block re-organization of the discrete equations can be exploited in a variety of ways to economize the solution process.

Of interest here is the fact that, although fast solvers [2–5] may not be appropriate or efficient for the original domain, D, they often can be used to advantage on the individual substructures, E_k. In particular, there has been much work recently on application of substructuring methods to the fast solution of low-order finite element approximations to elliptic problems on rectangularly decomposable domains [6–9]. In this note, we consider an extension of these ideas to the case of general high-order finite element approximations, such as high-order h-type techniques, spectral element methods [10], and p-type finite element schemes [11]. The algorithm presented combines (outer) static condensation with (inner) fast eigenfunction techniques [2, 12], to obtain an order-independent optimal operation count of $O(N^{5/2})$ for an $N \times N$ grid.

SPATIAL DISCRETIZATION

We consider here arbitrary-order finite-element solution of the constant-coefficient Helmholtz equation in two dimensions,

$$\nabla^2 u - \lambda^2 u = f \qquad \text{in } D \tag{1a}$$

$$u = u_B \qquad \text{on } \partial D, \tag{1b}$$

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on domains D decomposable into rectangular subdomains (or "super-elements") E_k, $D = \bigcup_k E_k$. The algorithms to be presented are most efficient in the case when one is interested in solving the same equation (i.e., same homogeneous operator) for many different forcing functions, f, and the methods are therefore especially appropriate for time-integration of constant-coefficient (in time) partial differential equations in which the spatial operators are of the form given in Eq. (1). Our particular interest is in solution of the incompressible Navier–Stokes equations, in which Poisson and Helmholtz equations arise in treatment of the pressure and viscous terms, respectively [10].

The spatial discretization proceeds by first expanding the unknown, u, and inhomogeneity, f, in each super-element E_k in standard tensor-product Lagrangian-interpolant finite element basis functions [10],

$$u^k(x, y) = u^k_{mn} h_m(x) h_n(y) \tag{2a}$$

$$f^k(x, y) = f^k_{mn} h_m(x) h_n(y), \tag{2b}$$

where summation from 0 to M over repeated indices is assumed. Here the $h_p(z)$ are Lagrangian interpolants, which may be of any order from unity (corresponding to piecewise linear interpolation, in which a super-element is made up of $M \times M$ bilinear finite elements) to M (corresponding to a subdomain which is a single spectral element [10]). For the sake of simplicity, we assume that the representation (2) is the same for all super-elements, and that, furthermore, all super-elements are identical squares with sides of length L. In practice, (2) can be replaced with any substructure-dependent representation which reduces to tensor-product form for the internal degrees-of-freedom.

We now insert (2) into the variational form for (1) [13], and require stationarity with respect to variations in the nodal values. This gives the following superelemental equations,

$$C^k_{ijmn} u^k_{mn} = B^k_{ijmn} f^k_{mn}, \tag{3}$$

where

$$C^k_{ijmn} = A^k_{ijmn} - \lambda^2 B^k_{ijmn}, \tag{4a}$$

and

$$A^k_{ijmn} = -\mathbf{A}_{im}\mathbf{B}_{jn} - \mathbf{A}_{jn}\mathbf{B}_{im} \tag{4b}$$

$$B^k_{ijmn} = \mathbf{B}_{im}\mathbf{B}_{jn}. \tag{4c}$$

Here $\mathbf{A}_{ij}$ and $\mathbf{B}_{ij}$ are (symmetric, positive-definite) finite element matrices corresponding to the one-dimensional Laplacian and mass operators on an interval

equivalent to one side of a super-element. For instance, for super-elements made up of M identical linear elements, $\mathbf{A}_{ij}$ and $\mathbf{B}_{ij}$ are tri-diagonal operators given by $\mathbf{A}_{i,i} = \gamma_i/\Delta z$, $\mathbf{A}_{i,i+1} = \mathbf{A}_{i,i-1} = -1/\Delta z$, $\mathbf{B}_{i,i} = (\gamma_i \, \Delta z/3)$, $\mathbf{B}_{i,i+1} = \mathbf{B}_{i,i-1} = (\Delta z/6)$. Here $\gamma_i = 1$ for $i = 0$, M, $\gamma_i = 2$ otherwise, and $\Delta z = L/M$. For the case of a super-element consisting of a single Mth-order Chebyshev spectral element, $\mathbf{A}_{ij}$ and $\mathbf{B}_{ij}$ are now full matrices, for which explicit formulae are given in [10].

Solution Algorithm

We now present the solution algorithm for the set of super-elemental equations given in Eqs. (3)–(4). The basic strategy is to use static condensation to decouple the super-elements [1], followed by fast eigenfunction (tensor-product) solution [2, 12] for the degrees-of-freedom interior to the individual subdomains. We first briefly describe the block elimination procedure, and then turn to the construction of the inner solvers.

Denoting the vector of nodes that lie on a given super-element boundary as $[^Bu^k]$, and the remaining interior nodes $[^Iu^k]$, the super-element equations can be written in block form as

$$[a^k][^Bu^k] + [b^k]^T[^Iu^k] = [^Bg^k] \tag{5a}$$

$$[b^k][^Bu^k] + [c^k][^Iu^k] = [^Ig^k], \tag{5b}$$

where

$$g_{ij}^k = B_{ijmn}^k f_{mn}^k. \tag{5c}$$

Standard block elimination then gives

$$[\bar{a}^k][^Bu^k] = [^B\bar{g}^k] \tag{6a}$$

$$[c^k][^Iu^k] = [^Ig^k] - [b^k][^Bu^k], \tag{6b}$$

where the statically-condensed matrix, $[\bar{a}^k]$, and associated inhomogeneity, $[^B\bar{g}^k]$, are given by

$$[\bar{a}^k] = [a^k] - [b^k]^T[c^k]^{-1}[b^k], \tag{7a}$$

and

$$[^B\bar{g}^k] = [^Bg^k] - [b^k]^T[c^k]^{-1}[^Ig^k], \tag{7b}$$

respectively. The global system for the statically-condensed nodes is then constructed from (7) by direct stiffness summation.

The key here is that the various super-elements are coupled only through the equations for the boundary nodes, $[^Bu^k]$. We therefore first solve the global system

derived from direct stiffness assembly of (6a) for the $[^B u^k]$. The problem defined on the individual super-elements for the remaining degrees-of-freedom (the $[^I u^k]$) is now a separable problem with homogeneous boundary conditions, for which a fast eigenfunction solver can be readily implemented. In particular, in (6)–(7) we require solution of

$$[c][v] = [h], \qquad (8)$$

which can be accomplished in $O(M^3)$ using a tensor-product method,

$$\bar{v}_{pq} = -(\varLambda_p + \varLambda_q + \lambda^2)^{-1} \sum_{j=1}^{M-1} \left(\sum_{i=1}^{M-1} h_{ij} \, \mathbf{S}_{ip} \right) \mathbf{S}_{jq} \qquad (9a)$$

$$v_{mn} = \sum_{q=1}^{M-1} \left(\sum_{p=1}^{M-1} \bar{v}_{pq} \mathbf{S}_{mp} \right) \mathbf{S}_{nq}. \qquad (9b)$$

Here $\mathbf{S}_{iq}$ and $\varLambda_q$ are the eigenvectors and eigenvalues, respectively, of the generalized eigenvalue problem

$$\mathbf{A}_{ij}\mathbf{S}_{jp} = \mathbf{B}_{ij}\mathbf{S}_{jp}\varLambda_p \qquad (10a)$$

$$\mathbf{S}_{0p} = \mathbf{S}_{Mp} = 0, \qquad (10b)$$

normalized by $\mathbf{S}_{mp}\mathbf{B}_{mn}\mathbf{S}_{nq} = \delta_{pq}$ (B-orthogonality and reality of $\varLambda_p$ following from symmetry of $\mathbf{A}$ and $\mathbf{B}$).

A somewhat faster approach to the solution of (8) is the use of an eigenfunction expansion in only one direction (say, x), with matrix inversion in the other direction [3]. In this case, (9) is replaced with

$$\bar{v}_{pn} = \sum_{j=1}^{M-1} (\bar{\mathbf{C}}^p)_{nj}^{-1} \left(\sum_{i=1}^{M-1} h_{ij} \mathbf{S}_{ip} \right) \qquad (11a)$$

$$v_{mn} = \sum_{p=1}^{M-1} \bar{v}_{pn} \mathbf{S}_{mp}, \qquad (11b)$$

where

$$\bar{\mathbf{C}}_{ij}^p = -\mathbf{A}_{ij} - (\lambda^2 + \varLambda_p) \, \mathbf{B}_{ij}. \qquad (12)$$

Note by use of the generalized eigenvalue problem (10), we need not make any assumptions concerning the commutation of $\mathbf{A}$ and $\mathbf{B}$ [2].

COMPUTATIONAL COMPLEXITY

We now examine the computational complexity of the solution algorithm described above. For this purpose, we consider the problem of a square domain, D,

made up of a regular array of $P \times P$ super-elements ($N_S = P^2$). (Similar, but more complicated, estimates are possible in the case of nonrectangular domains.) We define the number of points per domain side as $N + 1$, and the number of points per superelement side as $M + 1$, giving a relationship between N, M, and P of $M = N/P$.

Although efficient iterative techniques have been developed for the coupling matrices resulting from low-order substructuring schemes [6–9], for the high-order methods of interest here we consider the simpler case where the statically-condensed global system is solved by direct banded Gaussian elimination. Determining the internal degrees-of-freedom from (6b) using (9) or (11), the operation count for the entire system per right-hand side is thus given by

$$W(P) = \alpha P N^2 + \beta N^3 / P \qquad 1 \ll P \ll N, \qquad N \Rightarrow \infty \qquad (13)$$

where α and β are constants independent of N and P. (Although the static condensation algorithm obviously admits a high degree of parallelism, we do not discuss this here, and it is not reflected in the operation count given above.) The first term in (13) corresponds to the Gaussian elimination for the super-element boundary points ($O(NP)$ unknowns by bandwith $O(N)$), whereas the second term is due to the eigenfunction solver for the $[{}^\prime u^k]$ (9) or (11), the formation of the terms on the right-hand side of (6a) and (6b), and the factorized construction of $[g^k]$ in (5c). Note we have neglected here the pre-processing work associated with the formation of the global Cholesky decomposition ($O(N^3 P)$), as we are interested in an asymptotic operation count for many right-hand sides. (The pre-processing work associated with the diagonalization and inversion of the one-dimensional operators is negligible for the case studied here.)

It is clear from differentiation of the expression (13) with respect to P that a minimum in the work occurs at $P^* = (\beta N/\alpha)^{1/2}$ ($N \Rightarrow \infty$). This corresponds to an optimal operation count of $W^*(P^*) = 2(\alpha\beta)^{1/2} N^{5/2}$ ($N \Rightarrow \infty$), indicating that the scheme is significantly faster than, say, a direct banded solver or optimal SOR, both of which require $O(N^3)$ operations. Furthermore, this work estimate of $O(N^{5/2})$ is independent of the order of the scheme, valid for the whole range of approximations from second-order bilinear finite elements to exponentially-convergent spectral elements of order $M \sim O(N^{1/2})$ [10].

For the case of bilinear elements on a uniform mesh, the $O(N^{5/2})$ operation count is certainly not the best possible, as one can achieve an $O(N^2 \log N)$ count [4] by using the FFT algorithm on super-elements as large as possible (e.g., $P = 1$ in our square example above). However, such an option does not typically exist for variable-mesh or high-order schemes, in which case our $O(N^{5/2})$ estimate seems quite good. Furthermore, even for low-order discretizations, the current algorithm works for a larger class of problems than fast transform techniques.

It should be noted that, in the case of super-elements consisting of elements of fixed order (e.g., bicubic), the choice of P in no way effects the numerical discretization or the resulting accuracy of the solution. However, if one chooses to identify a single spectral element with each super-element, the order (and hence,

accuracy) of the scheme now scales with P, and the sense in which P^* is optimal is no longer as clear. A detailed description of application of the technique to spectral element solution of the Poisson and Navier–Stokes equations can be found in [14, 15], in which the operation counts given here are verified empirically, and the practical problems associated with optimal choice of P are addressed [14].

The methods and operation counts presented here are easily generalized to any situation in which the tensor-product or single-direction eigenfunction expansion methods can be applied to the subdomain problems. In particular, the one-dimensional operators, $\mathbf{A}_{ij}$, $\mathbf{B}_{ij}$, need not be the same for the two co-ordinate directions, allowing, for instance, for solution of elliptic problems in cylindrical coordinates. It is, nevertheless, a serious limitation that the method applies only to domains which can be represented as the sum of rectangular subdomains, as this excludes the important case of general isoparametric elements [16]. Furthermore, in three dimensions, even in regular domains the method is not competitive with techniques such as conjugate gradient iteration [17, 18], due to the relatively large number of subdomain boundary nodes in higher space dimensions.

CONCLUSIONS

We have presented here a direct fast solver for arbitrary-order finite and spectral element discretizations of separable elliptic equations on rectangularly decomposable domains. The method is based on substructure concepts, in which banded elimination is used on statically-condensed equations for subdomain boundary nodes, with subsequent fast eigenfunction solution of the decoupled subdomain problems. By minimization of computational work with respect to degree of domain subdivision, an order-independent optimal operation count of $O(N^{5/2})$ is obtained; due to significant pre-processing requirements, this estimate is only achieved in the limit of many solves per given homogeneous operator, such as in implicit solution in time of parabolic partial differential equations.

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